Linear Algebra I

16/12/2019, Monday, 19:00 – 21:00

You are **NOT** allowed to use any type of calculators.

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1 Systems of linear equations

(5 + 15 + 5 = 25 pts)

Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 + 2x_4 - x_5 &= 1 \\ 2x_1 + 2x_2 + 3x_3 + 6x_4 - 3x_5 &= 1 \\ 3x_1 + 3x_2 + 4x_3 + 8x_4 - 3x_5 &= 6 \\ x_1 + x_2 + 3x_3 + 6x_4 + 4x_5 &= a \end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are unknowns and a is a real number.

(a) Write down the augmented matrix.

(b) By performing elementary row operations, put the augmented matrix into **reduced** row echelon form.

(c) Determine the solution set.

2 Determinants

(10 + 5 = 15 pts)

Let M_n be the $n \times n$ matrix given by

$$M_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix},$$

i.e.,

$$M_1 = 1, \qquad M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

(a) Show that $\det(M_{k+1}) = \det(M_k)$ for all $k \geqslant 1$.

(b) Show that $\det(M_k) = 1$ for all $k \geqslant 1$.

3 Partitioned matrices (15 + 5 = 20 pts)

Let A, B, C be $n \times n$ matrices. Consider the matrix

$$M = \begin{bmatrix} A & B \\ C & I_n \end{bmatrix}$$

where I_n denotes the $n \times n$ identity matrix.

(a) Find necessary and sufficient conditions (in terms of A, B, C) under which M is nonsingular.

(b) Find the inverse of M whenever it is nonsingular.

4 Vector spaces (15 + 15 = 30 pts)

Let $\boldsymbol{x} \in \mathbb{R}^n$ and

$$S_{\boldsymbol{x}} = \{A \in \mathbb{R}^{n \times n} \mid \boldsymbol{x}^T A \boldsymbol{x} = 0\}.$$

(a) Show that $S_{\boldsymbol{x}}$ is a subspace of the vector space $\mathbb{R}^{n \times n}$.

(b) Let $n = 2$ and $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find a basis for $S_{\boldsymbol{x}}$ and determine its dimension.

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10 pts free